Entropic Hardness of Module-LWE from Module-NTRU

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INDOCRYPT'22 - Dec. 11th-14th, 2022

Hardness of Module Learning With Errors

Entropic Hardness of Module Learning With Errors

- with **General Secret Distributions** carrying sufficient **Entropy**,

Entropic Hardness of Module Learning With Errors from Module-NTRU

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- from the hardness of **Module-NTRU**,

Our Result (full version at https://ia.cr/2022/245)

Entropic Hardness of Module Learning With Errors from Module-NTRU

- with **General Secret Distributions** carrying sufficient **Entropy**,
- from the hardness of **Module-NTRU**,
- over **General Number Fields** in a **Rank-Preserving** reduction.

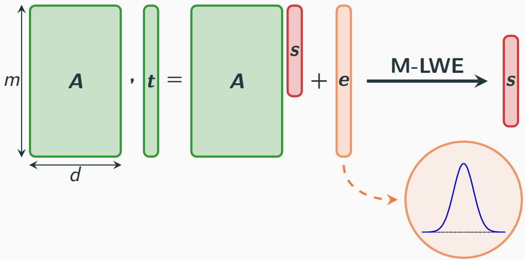
Entropic Hardness of Module Learning With Errors from Module-NTRU

- with **General Secret Distributions** carrying sufficient **Entropy**,
- from the hardness of **Module-NTRU**,
- over **General Number Fields** in a **Rank-Preserving** reduction.

Other Contributions:

- Improves on [BD20] (R-LWE) when rank is 1.
- Spectral analysis of multiplication matrices in general number fields (follow-up in [BJRW22] recently published at Journal of Cryptology).

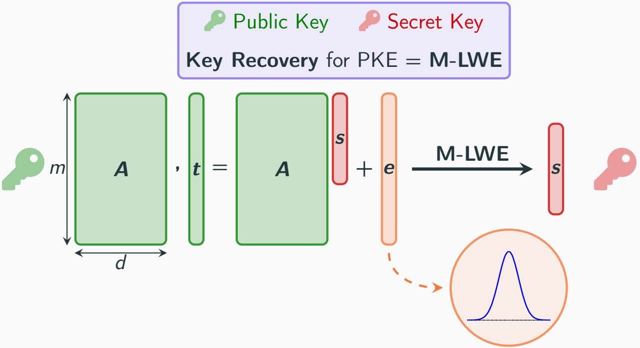
Module Learning With Errors (M-LWE)



where $\boldsymbol{A} \hookleftarrow \text{Unif}(\mathcal{R}_q^{m \times d})$, $\boldsymbol{s} \hookleftarrow \mathcal{S}$ (over $\mathcal{R}^d$), and $\boldsymbol{e} \hookleftarrow \text{Gauss}(\sigma_{\boldsymbol{e}})$.

$\mathcal{R}$: Ring of integers of a number field of degree n.
Typical choice: $\mathcal{R} = \mathbb{Z}[x]/\langle \Phi \rangle$, Φ a cyclotomic polynomial of degree n.

Parameterized by distribution $\mathcal{S}$. Later: **Entropy Requirements**



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Why M-LWE? NIST announced future PQC standards in July 2022.

Encryption	Signature
Crystals-Kyber	Crystals-Dilithium
	Falcon
	SPHINCS+

M-LWE-based (selected for CNSA Suite 2.0): Crystals-Kyber, Crystals-Dilithium

lattice-based: Crystals-Kyber, Crystals-Dilithium, Falcon

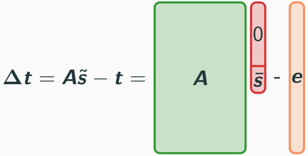
Why Entropic Hardness of M-LWE?

Why M-LWE? NIST announced future PQC standards in July 2022.

Why Entropic Hardness? Resilience against leakage. Example:

1. Physical attack to recover a noisy secret $\tilde{\boldsymbol{s}}$.

2. Target a new M-LWE instance



?

Under what condition on $\boldsymbol{s}'$ is the problem still hard?
$\boldsymbol{s}'$ must have enough **entropy** $\longrightarrow$ **Entropic hardness**

Intuition: Lossiness

$$H_\infty(\boldsymbol{s}' | \boldsymbol{A}, \boldsymbol{A}\boldsymbol{s}' + \boldsymbol{e}) \text{ large} \Longrightarrow \text{M-LWE instance with secret } \boldsymbol{s}' \text{ hard}$$

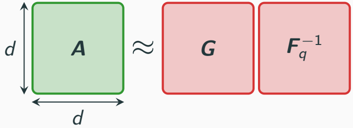
NTRU

$$a \approx g/f$$

$a \sim \mathrm{Unif}(\mathcal{R}_q)$, $f, g \sim \mathrm{Gauss}(\mathcal{R}, \gamma)$

↓

(square) **M-NTRU**



$\boldsymbol{A} \sim \mathrm{Unif}(\mathcal{R}_q^{d \times d})$, $\boldsymbol{F}, \boldsymbol{G} \sim \mathrm{Gauss}(\mathcal{R}^{d \times d}, \gamma)$

NTRU

$$a \approx g/f$$

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(square) **M-NTRU**

$$\boldsymbol{A} \approx \boldsymbol{G}\,\boldsymbol{F}_q^{-1} \quad (d \times d)$$

$\boldsymbol{A} \sim \mathsf{Unif}(\mathcal{R}_q^{d\times d}),\ \boldsymbol{F}, \boldsymbol{G} \sim \mathsf{Gauss}(\mathcal{R}^{d\times d}, \gamma)$

Randomized **NTRU**

(with HNF-R-LWE) [BD20]

$$\boldsymbol{a} \approx \boldsymbol{e} \cdot (g/f) + \boldsymbol{e}' \quad (m)$$

$\boldsymbol{e}, \boldsymbol{e}' \sim \mathsf{Gauss}(\mathcal{R}^m, \alpha)$

Randomized (square) **M-NTRU**

(with HNF-M-LWE)

$$\boldsymbol{A} \approx \boldsymbol{E}\,\boldsymbol{G}\,\boldsymbol{F}_q^{-1} + \boldsymbol{E}'$$

$\boldsymbol{E}, \boldsymbol{E}' \sim \mathsf{Gauss}(\mathcal{R}^{m\times d}, \alpha)$

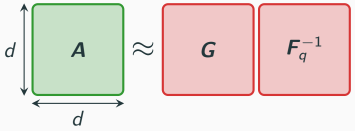
What About Module-NTRU?

NTRU

$$a \approx g/f$$

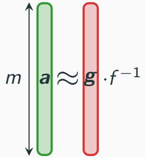
$a \sim \mathrm{Unif}(\mathcal{R}_q),\ f, g \sim \mathrm{Gauss}(\mathcal{R}, \gamma)$

(square) M-NTRU



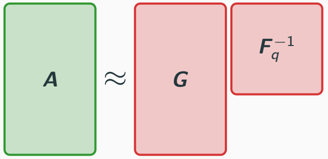
$\boldsymbol{A} \sim \mathrm{Unif}(\mathcal{R}_q^{d \times d}),\ \boldsymbol{F}, \boldsymbol{G} \sim \mathrm{Gauss}(\mathcal{R}^{d \times d}, \gamma)$

Multi-Key NTRU



$\boldsymbol{g} \sim \mathrm{Gauss}(\mathcal{R}^m, \gamma), f \sim \mathrm{Gauss}(\mathcal{R}, \gamma)$

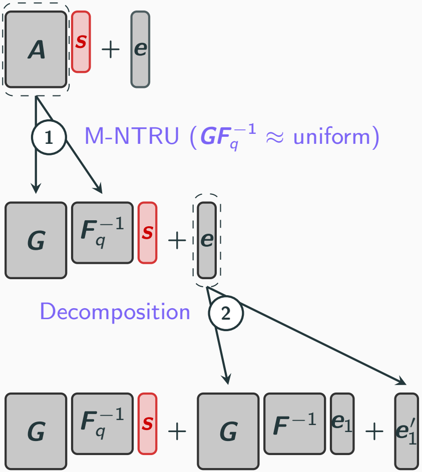
(rectangular) M-NTRU



$\boldsymbol{G} \sim \mathrm{Gauss}(\mathcal{R}^{m \times d}, \gamma), \boldsymbol{F} \sim \mathrm{Gauss}(\mathcal{R}^{d \times d}, \gamma)$

Entropic Hardness of M-LWE from M-NTRU

Replacing $\boldsymbol{A}$ by $\boldsymbol{G}\boldsymbol{F}_q^{-1}$, with $\boldsymbol{F}, \boldsymbol{G}$ Gaussian and $\boldsymbol{F}_q^{-1} = (\boldsymbol{F} \bmod q\mathcal{R})^{-1}$.
The secret $\boldsymbol{s}$ is only assumed to have **large enough entropy**.



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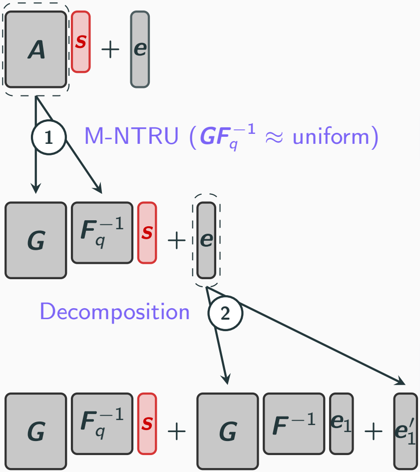

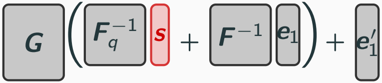

$$
\begin{aligned}
& H_\infty(\boldsymbol{s} \mid \boldsymbol{G}(\boldsymbol{F}_q^{-1}\boldsymbol{s} + \boldsymbol{F}^{-1}\boldsymbol{e}_1) + \boldsymbol{e}_1') \\
\geq\ & H_\infty(\boldsymbol{s} \mid \boldsymbol{F}_q^{-1}\boldsymbol{s} + \boldsymbol{F}^{-1}\boldsymbol{e}_1) \\
\geq\ & H_\infty(\boldsymbol{s} \mid \boldsymbol{F}_q^{-1}\boldsymbol{s} + \boldsymbol{F}^{-1}\boldsymbol{e}_2) \qquad (\boldsymbol{e}_2 \in \mathcal{L}(\boldsymbol{F})) \\
=\ & H_\infty(\boldsymbol{s} \mid \boldsymbol{s} + \boldsymbol{e}_2) \\
\geq\ & H_\infty(\boldsymbol{s} \mid \boldsymbol{s} + \boldsymbol{e}') - nd \log_2 \|\boldsymbol{F}\|_2 \\
\geq\ & H_\infty(\boldsymbol{s}) - nd \log_2 \frac{q}{\sigma_{\boldsymbol{e}'}} - nd \log_2 \|\boldsymbol{F}\|_2
\end{aligned}
$$

(2) $\sigma_{\boldsymbol{e}} > \sigma_{\boldsymbol{e}'} \|\boldsymbol{G}\boldsymbol{F}^{-1}\|_2$

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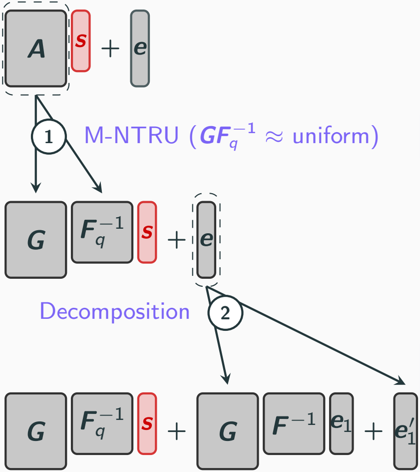

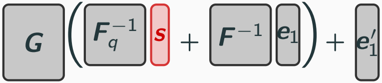

$$H_\infty(\boldsymbol{s} | \boldsymbol{G}(\boldsymbol{F}_q^{-1}\boldsymbol{s} + \boldsymbol{F}^{-1}\boldsymbol{e}_1) + \boldsymbol{e}_1')$$

$$\geq H_\infty(\boldsymbol{s} | \boldsymbol{F}_q^{-1}\boldsymbol{s} + \boldsymbol{F}^{-1}\boldsymbol{e}_1)$$

$$\geq H_\infty(\boldsymbol{s} | \boldsymbol{F}_q^{-1}\boldsymbol{s} + \boldsymbol{F}^{-1}\boldsymbol{e}_2) \qquad (\boldsymbol{e}_2 \in \mathcal{L}(\boldsymbol{F}))$$

$$= H_\infty(\boldsymbol{s} | \boldsymbol{s} + \boldsymbol{e}_2)$$

$$\geq H_\infty(\boldsymbol{s} | \boldsymbol{s} + \boldsymbol{e}') - nd \log_2 \|\boldsymbol{F}\|_2$$

$$\geq H_\infty(\boldsymbol{s}) - nd \log_2 \frac{q}{\sigma_{\boldsymbol{e}'}} - nd \log_2 \|\boldsymbol{F}\|_2$$

(2) $\sigma_{\boldsymbol{e}} > \sigma_{\boldsymbol{e}'} \|\boldsymbol{G}\boldsymbol{F}^{-1}\|_2$

Singular Values to optimize

Our contribution

- ✓ Reduction from Module-NTRU to Module-LWE with **general**[1] **secret distributions**.

Related Work

- Other reduction in [LWW20] from Module-LWE (uniform secret) to Module-LWE (general secret).
  - ✗ Not rank-preserving.
  - ✓ Assumption proven on module lattices.
  - = Parameter regimes with sometimes better or worse results.

Open Questions

- ? Reduction from module lattice problems to Module-NTRU?
- ? Prove the hardness of Module-LWE with low-entropy secret distributions without increasing the rank?

[1] with some restrictions though

Thank you for your attention!

Questions?

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