On the Hardness of Module Learning With Errors with Short Distributions

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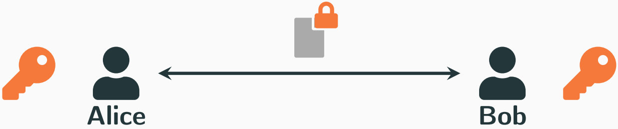
Seminaire Algo - Oct. 11th, 2022

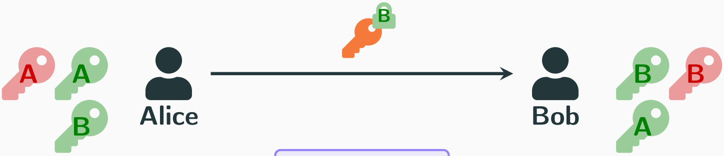
Reminder: Symmetric and Asymmetric Cryptography

Symmetric Cryptography



Alice and Bob must agree on the same key

Asymmetric Cryptography



Public Key
Secret Key

The Need For Post-Quantum Cryptography

What if we had a **Cryptographically Relevant Quantum Computer**[1]?

- **Quadratic quantum speed-up** with **Grover**'s algorithm [Gro96]: exhaustive key search of 🔑 in $O(\sqrt{\#\text{key space}})$;
- **Exponential quantum speed-up** with **Shor**'s algorithm [Sho97]: factoring and discrete logarithm in $poly(\log n) \implies$ 🔓

The underlying hardness assumptions of modern cryptography (RSA, ECC) would no longer be valid.

Need: Design new cryptosystems from new mathematical problems that are hard to solve, even by a CRQC. And fast...

[1] NSA FAQ on Quantum Computing and Post-Quantum Cryptography

Potential Candidates: NIST PQC Standardization

NIST **PQC standardization process** launched in 2016. First round of standardized algorithms announced in July 2022:

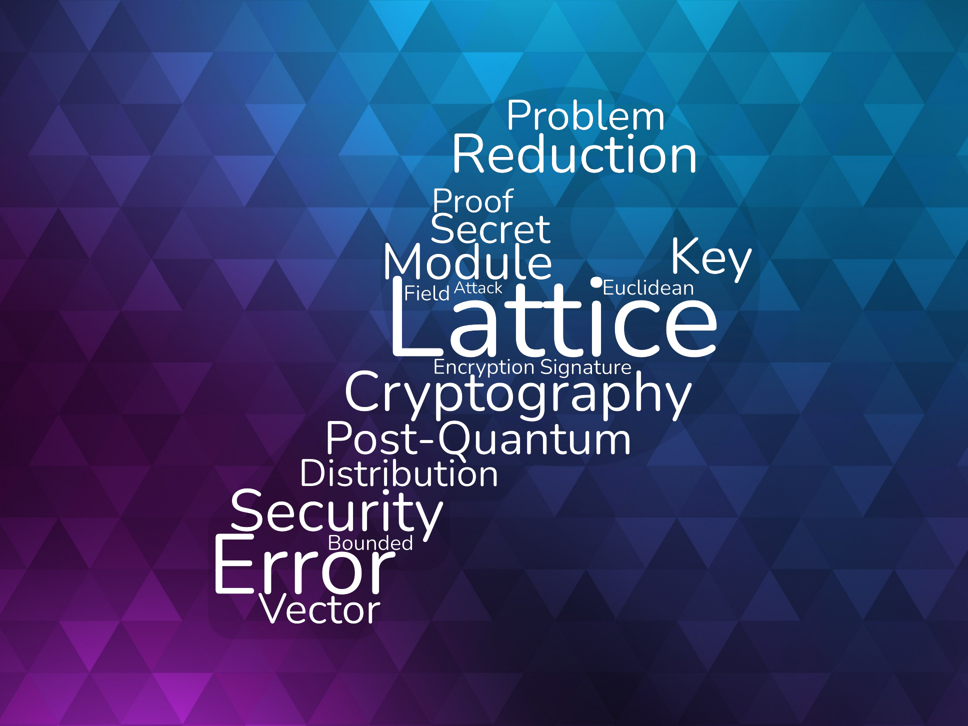
Encryption	Signature
Crystals-Kyber	Crystals-Dilithium
	Falcon
	SPHINCS+

M-LWE

lattice-based

NSA has already announced its CNSA Suite 2.0 for Quantum-Resistant algorithms. It includes **Kyber** and **Dilithium**.

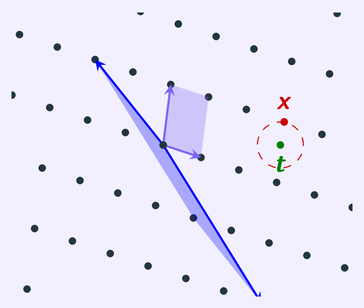
How robust is Module Learning With Errors with such short distributions? **Let's see**


Problem
Reduction
Proof
Secret
Module
Key
Field
Attack
Euclidean
Lattice
Encryption
Signature
Cryptography
Post-Quantum
Distribution
Security
Bounded
Error
Vector

Euclidean Lattice

$$\mathcal{L} = \left\{ \boxed{B} \boxed{x} \; ; \; x \in \mathbb{Z}^n \right\}$$

with basis $B \in \mathbb{R}^{n \times n}$.



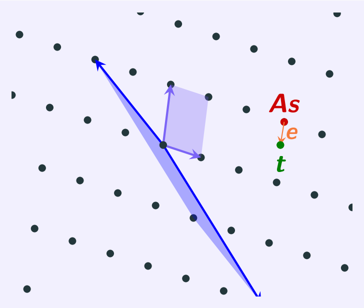
CVP$_t$ Given a target t, find $x \in \mathcal{L}$ that minimizes $\|x - t\|$.

You Said Lattice?

Euclidean Lattice

$$\mathcal{L} = \left\{ \boxed{\boldsymbol{B}} \boxed{\boldsymbol{x}} \; ; \; \boldsymbol{x} \in \mathbb{Z}^n \right\}$$

with basis $\boldsymbol{B} \in \mathbb{R}^{n \times n}$.



CVP$_t$ Given a target $\boldsymbol{t}$, find $\boldsymbol{x} \in \mathcal{L}$ that minimizes $\|\boldsymbol{x} - \boldsymbol{t}\|$.

Given $\boldsymbol{A} \in \mathbb{Z}_q^{m \times d}$ describing the lattice

$$\mathcal{L}_q(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{Z}^m : \exists \boldsymbol{s} \in \mathbb{Z}_q^d, \boldsymbol{A}\boldsymbol{s} = \boldsymbol{x} \bmod q\}$$

and $\boldsymbol{t} = \boldsymbol{A}\boldsymbol{s} + \boldsymbol{e} \bmod q$, solve **CVP$_t$** on $\mathcal{L}_q(\boldsymbol{A})$. This is **LWE**!

Set $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ for some integer q.

$$\left(\underbrace{\boldsymbol{A}}_{m \times d},\ \boldsymbol{t} = \boldsymbol{A}\boldsymbol{s} + \boldsymbol{e} \right) \xrightarrow{\textbf{LWE}^2} \boldsymbol{s}$$

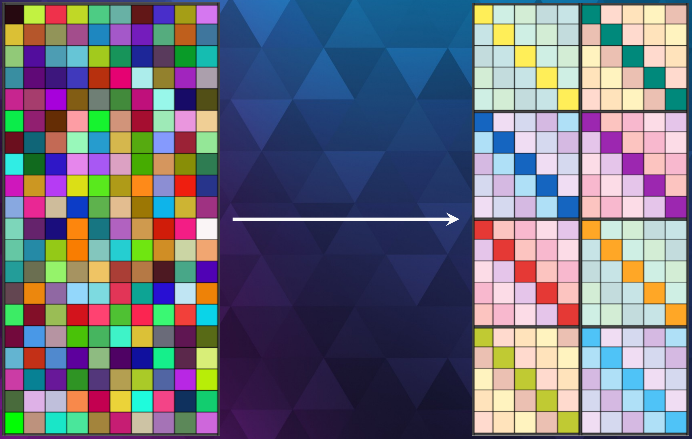
Public Key
Secret Key

where $\boldsymbol{A} \hookleftarrow \mathrm{Unif}(\mathbb{Z}_q^{m \times d})$, $\boldsymbol{s} \hookleftarrow \mathcal{D}_s$ (over $\mathbb{Z}^d$), and $\boldsymbol{e} \hookleftarrow \mathcal{D}_e$ (over $\mathbb{Z}^m$).

Standard [Reg05]:	$\mathcal{D}_s = \mathrm{Unif}(\mathbb{Z}_q^d)$	$\mathcal{D}_e = \mathrm{Gauss}(\mathbb{Z}^m)$
Binary Secret [BLP+13]:	$\mathcal{D}_s = \mathrm{Unif}(\{0,1\}^d)$	$\mathcal{D}_e = \mathrm{Gauss}(\mathbb{Z}^m)$
Binary Error [MP13]:	$\mathcal{D}_s = \mathrm{Unif}(\mathbb{Z}_q^d)$	$\mathcal{D}_e = \mathrm{Unif}(\{0,1\}^m)$

[2] The decision problem is to distinguish such $\boldsymbol{t}$ from $\mathrm{Unif}(\mathbb{Z}_q^m)$

Reduce needed storage and speed-up computations by adding Structure



Adding an Algebraic Structure for More Efficiency

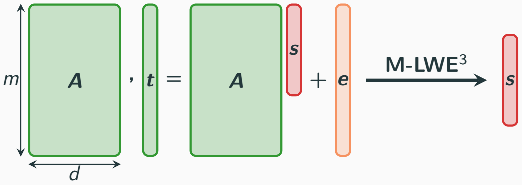
Replace $\mathbb{Z}$ with a ring $\mathcal{R} = \mathbb{Z}[x]/\langle f(x)\rangle$, e.g., $f(x) = x^n + 1$ with $n = 2^\ell$ and $\mathbb{Z}_q$ by $\mathcal{R}_q = \mathbb{Z}_q[x]/\langle f(x)\rangle$

$$\sum_{i=0}^{n-1} a_i \cdot x^i \in \mathcal{R} \xleftrightarrow{\text{embedding}} \begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix} \in \mathbb{Z}^n$$

$$\left(\sum_{i=0}^{n-1} a_i \cdot x^i\right) \cdot \left(\sum_{i=0}^{n-1} b_i \cdot x^i\right) \longleftrightarrow \begin{bmatrix} \mathrm{Rot}(\boldsymbol{a}) \end{bmatrix} \cdot \begin{bmatrix} b_0 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

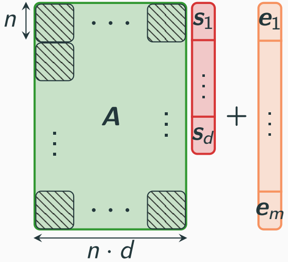
Efficiency: FFT-like algorithms, use of structured matrices.
Storage: Structured matrices represented by a single vector.

Module Learning With Errors as Structured LWE



where $\boldsymbol{A} \hookleftarrow \mathrm{Unif}(\mathcal{R}_q^{m\times d})$, $\boldsymbol{s} \hookleftarrow \mathcal{D}_s$ (over $\mathcal{R}^d$), and $\boldsymbol{e} \hookleftarrow \mathcal{D}_e$ (over $\mathcal{R}^m$).

A good choice would be over $S_1 = \{0,1\}[x]/\langle x^n + 1\rangle$.



Structured version of LWE
in dimensions nm & nd

[3]The decision problem is to distinguish such $\boldsymbol{t}$ from $\mathrm{Unif}(\mathcal{R}_q^m)$

What do we know so far?

Distributions	LWE	M-LWE
$\mathcal{D}_s = \mathrm{Unif}(\mathcal{R}_q^d)$ $\mathcal{D}_e = \mathrm{Gauss}(\mathcal{R}^m)$	[Reg05] [BLP+13]	[LS15] ?
$\mathcal{D}_s = \mathrm{Unif}(S_1^d)$ $\mathcal{D}_e = \mathrm{Gauss}(\mathcal{R}^m)$	[GKPV10] [BLP+13] [Mic18]	? ? ?
$\mathcal{D}_s = \mathrm{Unif}(\mathcal{R}_q^d)$ $\mathcal{D}_e = \mathrm{Unif}(S_1^m)$	[MP13]	?
$\mathcal{D}_s$ arbitrary $\mathcal{D}_e = \mathrm{Gauss}(\mathcal{R}^m)$	[BD20a] [BD20b] (R-LWE)	[LWW20] ?

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$\mathcal{D}_s = \mathrm{Unif}(\mathcal{R}_q^d)$ $\mathcal{D}_e = \mathrm{Gauss}(\mathcal{R}^m)$	[Reg05] [BLP+13]	[LS15] ❹ [BJRW20]
$\mathcal{D}_s = \mathrm{Unif}(S_1^d)$ $\mathcal{D}_e = \mathrm{Gauss}(\mathcal{R}^m)$	[GKPV10] [BLP+13] [Mic18]	❶ [BJRW20] ❷ [BJRW21] ?
$\mathcal{D}_s = \mathrm{Unif}(\mathcal{R}_q^d)$ $\mathcal{D}_e = \mathrm{Unif}(S_1^m)$	[MP13]	❸ [BJRW22b]
$\mathcal{D}_s$ arbitrary $\mathcal{D}_e = \mathrm{Gauss}(\mathcal{R}^m)$	[BD20a] [BD20b] (R-LWE)	[LWW20] ❺ [BJRW22a]

Today

❶ M-LWE is still hard with **small** $\boldsymbol{s}$ and **Gaussian** $\boldsymbol{e}$;

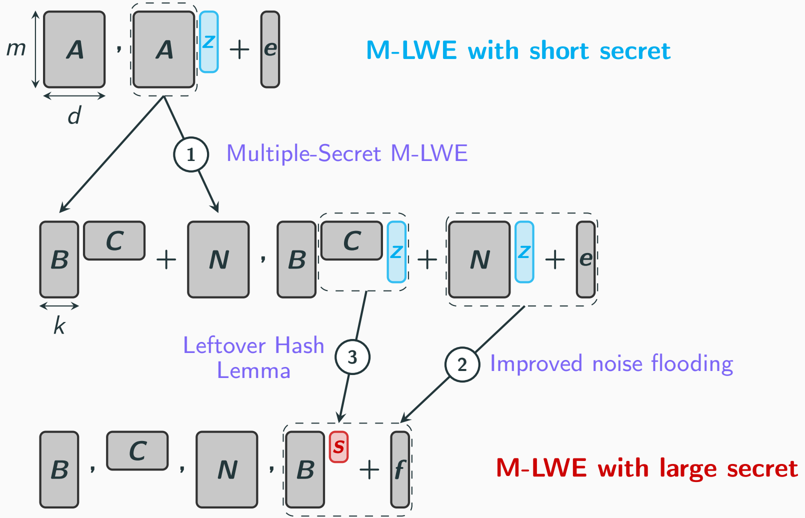
❷ Decisional M-LWE is still hard with **small** $\boldsymbol{s}$ and **Gaussian** $\boldsymbol{e}$;

❸ M-LWE is still hard with **small** d and $\boldsymbol{e}$, if m is not too large.

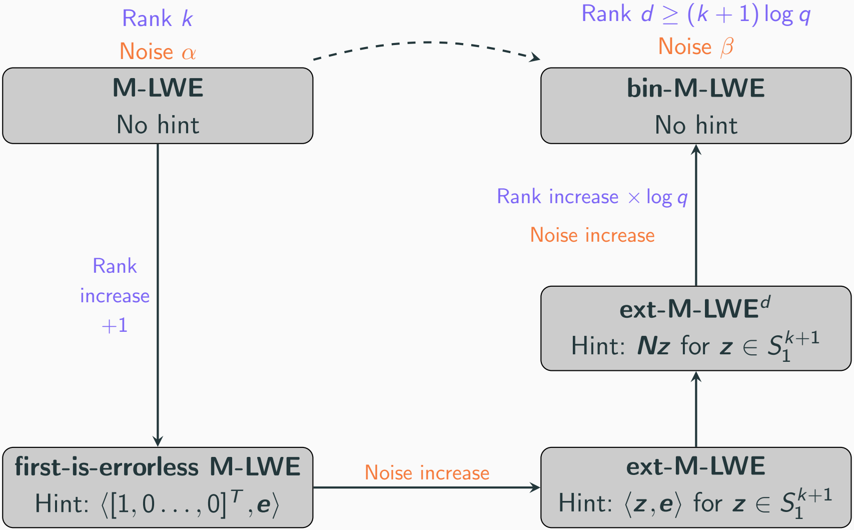
And now…

❶ Computational Hardness of M-LWE with Short Secret

The secret $\boldsymbol{z}$ is small (S_1^d) and the secret $\boldsymbol{s}$ is large ($\mathcal{R}_q^k$).

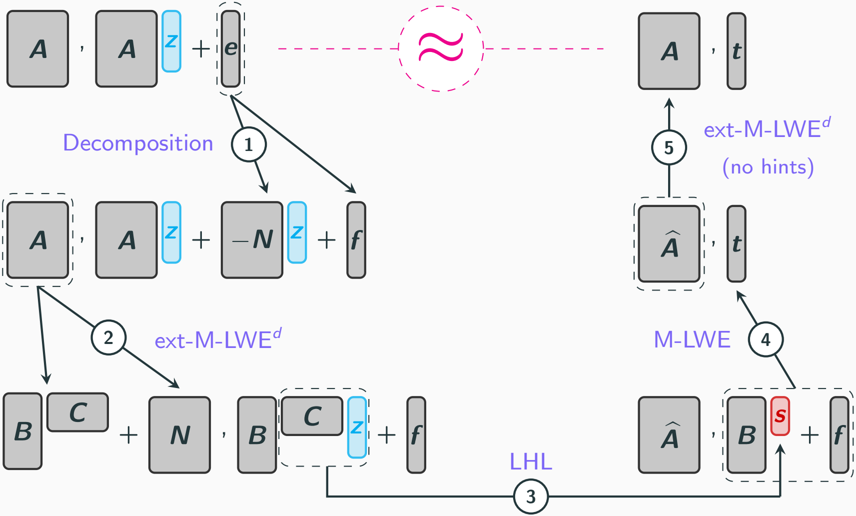


❷ Pseudorandomness of M-LWE with Short Secret (1/2)



❷ Pseudorandomness of M-LWE with Short Secret (2/2)

The secret z is small (S_1^d) and the secret s is large ($\mathcal{R}_q^k$).



$$\text{Standard M-LWE} \xrightarrow{\text{Reduction}} \text{Short Secret M-LWE}$$

Standard M-LWE	Short Secret M-LWE
modulus q	modulus q
ring degree n	ring degree n
secret $\mathbf{s} \in \mathcal{R}_q^k$	secret $\mathbf{z} \in S_1^d$
Gaussian width α	Gaussian width β
rank k	rank d

Property	Contribution ❶	Contribution ❷
Minimal rank d	$k \log q + \Omega(\log n)$	$(k+1) \log q + \omega(\log n)$
Noise ratio β/α	$O(n^2\sqrt{md})$	$O(n^2\sqrt{d})$
Conditions on q	prime	other restrictions[4]
Decision/Search	search	decision

Both proofs have their (dis)advantages

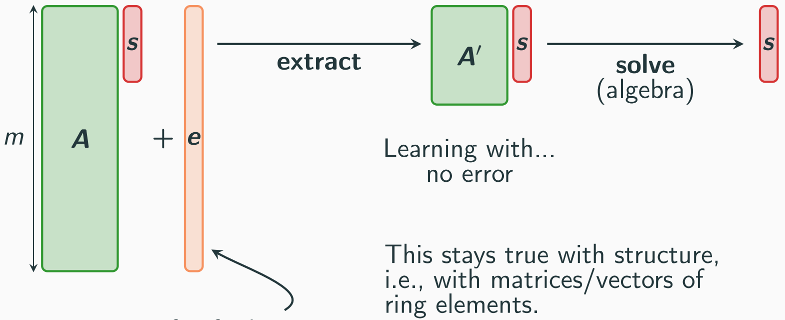
[4] In power-of-two cyclotomic fields, q must be prime such that $q = 5 \bmod 8$.

❸ Computational Hardness of M-LWE With Short Error

Idea: Prove that $(\boldsymbol{s}, \boldsymbol{e}) \mapsto \boldsymbol{A}\boldsymbol{s} + \boldsymbol{e}$ is one-way when $\boldsymbol{e}$ has small uniform coefficients. Reason on the dual function $\boldsymbol{e} \mapsto \boldsymbol{B}^T\boldsymbol{e}$.

Uninvertible is not enough.

Result: It is one-way if $\boldsymbol{A}$ is not too tall, i.e., m not too large. Why?



This stays true with structure, i.e., with matrices/vectors of ring elements.

Lots of 0 if $\boldsymbol{e}$ has small coefficients

Wrapping Up

Our contributions

- **Hardness** of a main problem, with (close to) **practical parameters**.

Lattice-based Cryptography

- **Most promising PQC successor** of RSA/ECC.
- **Mathematical problems on lattices** that are (confidently assumed) hard to solve even for a quantum computer.

What's next?

- Keep **closing the gap** between provably secure parameter sets and the ones used in practice (small ones).
- **Use** these stretched assumptions to **design efficient PQC schemes** (done, see NIST) with additional features (ok there is still work to do).

Thank you for your attention!

Questions?

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