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RESILIENCE

SESSION ID: **CRYP-W13C**

On the Hardness of Module-LWE with Binary Secrets

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Our Result (https://ia.cr/2021/265)

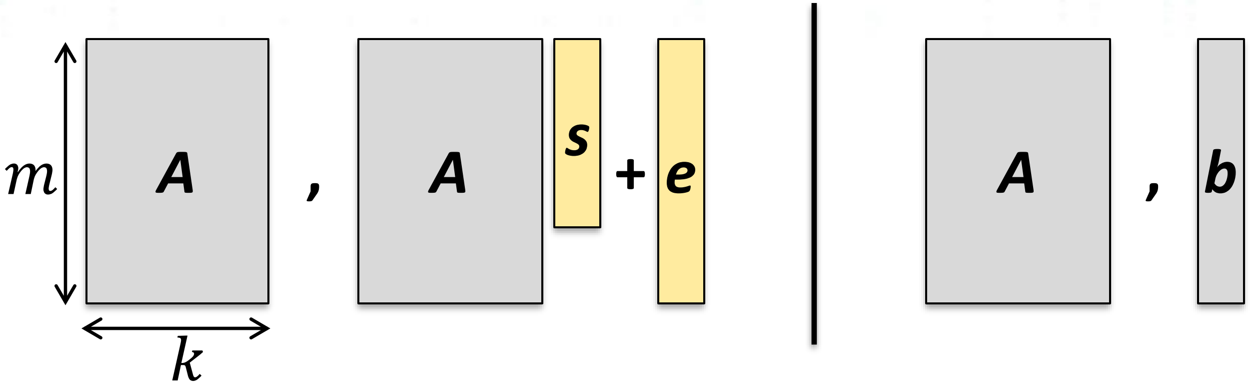
We (im)prove the theoretical hardness of **Module Learning With Errors** with **Binary Secrets**

- Over **cyclotomic fields** (degree n)
- For a **super-logarithmic module rank**: $d = \omega(\log n)$
- Down to **linearly small modulus**: $q \geq 2n$
- With a **small noise increase**: $\beta = \alpha \cdot \Theta(n^2\sqrt{d})$

We reduce the gap between **theoretical** and **practical** hardness when using small secrets

Module Learning With Errors (M-LWE)

The M-LWE problem asks to distinguish between two cases:



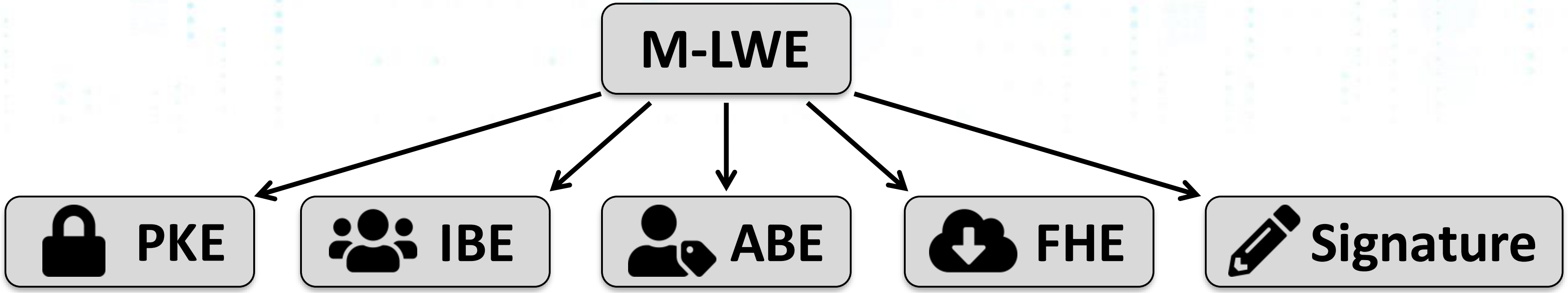
where $\boldsymbol{A} \hookleftarrow U(R_q^{k\times m})$, $\boldsymbol{s} \hookleftarrow U(R_q^k)$, $\boldsymbol{e} \hookleftarrow D_{R,\alpha q}^m$, and $\boldsymbol{b} \hookleftarrow U\left(R_q^m\right)$

$R = \mathbb{Z}[x]/\langle\Phi(x)\rangle$ is a cyclotomic ring with $\deg(\Phi) = n$. A popular choice is $n = 2^\ell$ yielding $\Phi(x) = x^n + 1$. We work in $R_q = \mathbb{Z}_q[x]/\langle\Phi(x)\rangle$.

Binary Secrets: $\boldsymbol{s}$ chosen from $R_2^k = (\mathbb{Z}_2[x]/\langle\Phi(x)\rangle)^k$

Edge cases: LWE ($n = 1 \Rightarrow R = \mathbb{Z}$) and R-LWE ($k = 1$)

Apply Module-LWE, Why Do We Care?



Key Encapsulation Mechanisms

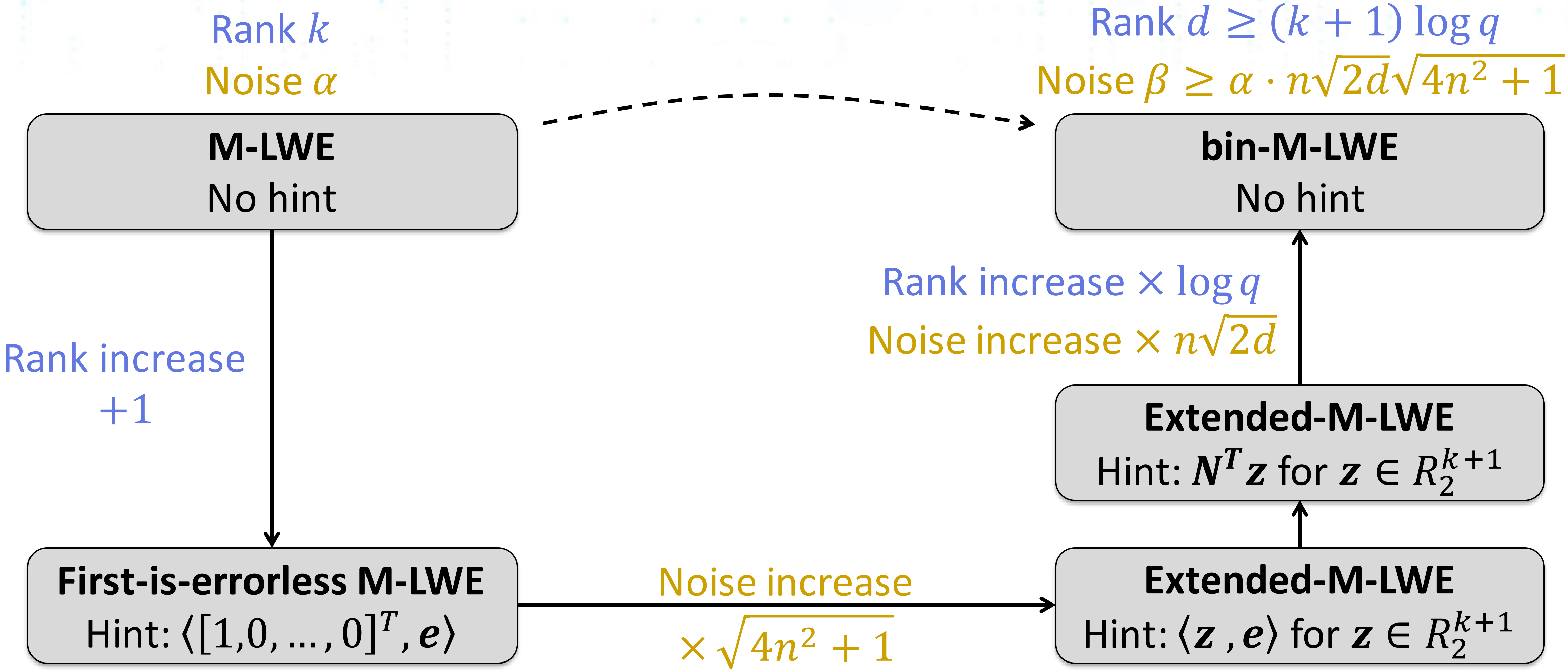
- **CRYSTALS-KYBER** [BDK+18]: based on Module-LWE
- **SABER** [DKRV18]: based on Module-LWR (deterministic)

Signature Schemes

- **CRYSTALS-DILITHIUM** [DKL+18]: based on Module-LWE

"*In NIST's current view, these structured lattice schemes appear to be the most promising general-purpose algorithms for public-key encryption/KEM and digital signature schemes*.", Third Round Candidate Announcement, July 22, 2020

Proof Structure following [BLP+13]



First-is-errorless M-LWE to Extended M-LWE: Construction

Reduction from first-is-errorless M-LWE to ext-M-LWE requires to construct, for any given $\boldsymbol{z} \in R_2^d$, a matrix $\boldsymbol{U}_{\boldsymbol{z}}$ such that

- $\boldsymbol{U}_{\boldsymbol{z}}$ is invertible in R_q
- $(\boldsymbol{U}_{\boldsymbol{z}}^{\perp})^T \boldsymbol{z} = \boldsymbol{0}$
- with minimal spectral norm (characterizes the noise growth)

$$\boldsymbol{z} = [z_1, \ldots, z_d]^T \in R_2^d$$

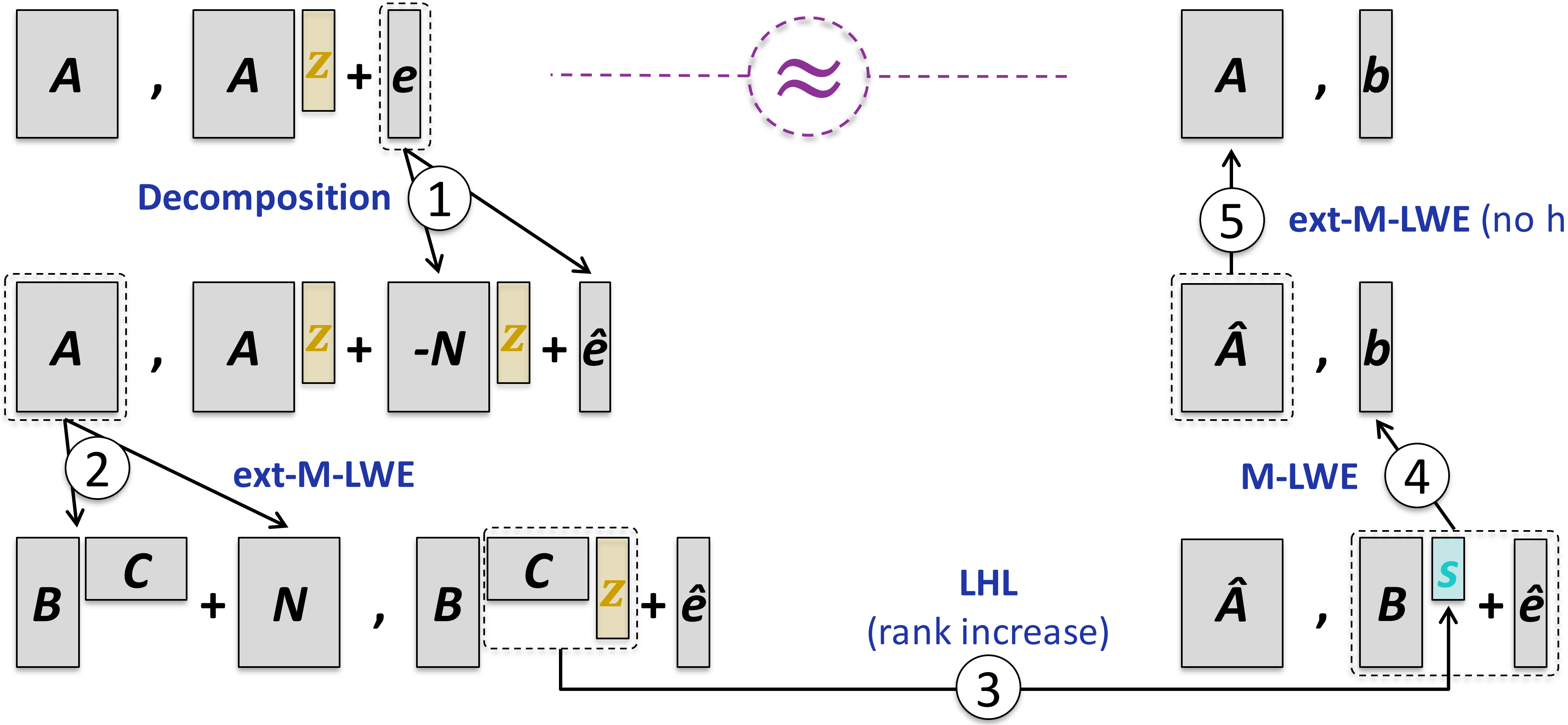
$$\boldsymbol{U}_{\boldsymbol{z}} = \begin{bmatrix} 1 & -z_2 & & \\ & z_1 & \ddots & \\ & & \ddots & -z_d \\ & & & z_{d-1} \end{bmatrix} \in R^{d \times d}$$

$\boldsymbol{U}_{\boldsymbol{z}}^{\perp}$

- ✔ Invertibility: restriction on q [LS18]
- ✔ Orthogonality: trivial
- ✔ Spectral norm: $\leq 2n$

Reduction to bin-M-LWE: Lossy Argument

Lossy argument: replacing $\boldsymbol{A}$ by $\hat{\boldsymbol{A}} = \boldsymbol{BC} + \boldsymbol{N}$. The secret $\boldsymbol{z}$ is binary and the secret $\boldsymbol{s}$ is modulo q.



Conclusion

Related Work

- Setting $n = 1$ yields the result from [BLP+13]
- Our previous reduction [BJRW20] achieves similar rank d and modulus q, but larger noise growth $\beta/\alpha = \Theta(n^2 d\sqrt{m})$. We improve it by a factor of $\sqrt{md}$

Open Problems

- Smaller ranks: rank $d = 1$ (R-LWE)
- Other number fields than cyclotomics

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Thank You!

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